FEW SMOOTH d-POLYTOPES WITH N LATTICE POINTS

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Abstract. We prove that, for fixed N there exist only finitely many embeddings of $\mathbb{Q}$-factorial toric varieties X into $\mathbb{P}^N$ that are induced by a complete linear system. The proof is based on a combinatorial result that for fixed nonnegative integers d and N, there are only finitely many smooth d-polytopes with N lattice points. The argument is turned into an algorithm to classify smooth 3-polytopes with ≤ 12 lattice points.

1. Introduction

The present note has two target audiences: combinatorialists and algebraic geometers. We give combinatorial proofs of results motivated by the algebraic geometry of toric varieties. We provide two introductions with statements of the main results in the language of divisors on toric manifolds on the one hand, and in the language of lattice polytopes on the other. In Section 2, we collect the relevant entries from the dictionary translating between the two worlds. In Section 3 we prove our theorems, and in Section 4 we report on first classification results.

1.1. Introduction (for algebraic geometers).

The purpose of this note is to show that there are very few embeddings of $\mathbb{Q}$-factorial toric varieties[1] into projective space that are induced by a complete linear series.

Theorem 1. *Let N be a nonnegative integer. Then there exist only finitely many embeddings of $\mathbb{Q}$-factorial toric varieties X into $\mathbb{P}^N$ that are induced by a complete linear series.*

Note however, that every Hirzebruch surface $\mathbb{F}_a$ admits an embedding into $\mathbb{P}^5$ (see Example 17), so we cannot omit the hypothesis that the embedding is induced by a complete linear series. Moreover, for non-$\mathbb{Q}$-factorial toric varieties the embedding dimension does not even bound the degree, see Example 15.

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[1] All toric varieties appearing in this paper are normal by construction.

Using Oda's classification of smooth 3-dimensional toric varieties that are minimal with respect to equivariant blow ups, we classify all 3-dimensional embeddings of smooth toric varieties into $\mathbb{P}^{\leq 11}$ using a complete linear series. In the appendix we present the complete list of the corresponding 3-polytopes with ≤ 12 lattice points up to equivalence (see Definition 3).

One possible application of this classification is a long standing open question by Oda [Oda08] whether every embedding of a smooth toric variety that is induced by a complete linear series is projectively normal. If we require our embedding to be projectively normal, then standard arguments from algebraic enumerative combinatorics imply finiteness (see Remark 14).

There is an entire hierarchy of successively stronger conjectures concerning embeddings of smooth projective toric varieties which are open even in dimension 3, (compare [MFO07, p.2313]). The principal obstacle to theoretical progress on Oda's question and the related conjectures is a serious lack of well understood examples. Recently Gubeladze [Gub09] has shown that any lattice polytope with sufficiently long edges (depending on the dimension) gives rise to a projectively normal embedding. In view of this result, if there exists a counterexample, it is more likely to be a small polytope. Yet, all polytopes in our classification up to 12 lattice points satisfy the even strongest of these conjectures (see Corollary 24).

1.2. **Introduction (for polytope people).** The purpose of this note is to show that there are very few lattice polytopes with a unimodular normal fan. More generally, we prove the following.

Theorem 2. *Let N be a nonnegative integer, and let $\mathcal{F}$ be a finite family of rational cones. Then there are only finitely many lattice polytopes with $N+1$ lattice points all whose normal cones are integrally equivalent to cones belonging to $\mathcal{F}$.*

In this theorem it would not be enough to fix the combinatorial type of the polytope (see Examples 15 & 16). The relevance of this result in algebraic geometry stems from the fact that a lattice polytope corresponds to an ample line bundle on a projective toric variety (see Section 2).

Oda classified 3-dimensional unimodular fans with ≤ 8 rays that are minimal with respect to stellar subdivisions. Based on this classification, we classify all 3-dimensional polytopes with unimodular normal fan and ≤ 12 lattice points up to equivalence (see Definition 3). They are listed in the appendix.

One possible application of this classification is a long standing open question by Oda [Oda08] whether every polytope P with a unimodular normal fan is integrally closed: for $k \in \mathbb{Z}_{\geq 2}$ every lattice point in

the dilation kP should be the sum of k lattice points in P. If we require P to be integrally closed, then standard arguments from algebraic enumerative combinatorics imply finiteness (see Remark 14).

There is an entire hierarchy of successively stronger conjectures concerning polytopes with unimodular normal fan which are open, even in dimension 3. (Compare [MFO07, p.2313].) The principal obstacle to theoretical progress on Oda's question and the related conjectures is a serious lack of well understood examples. Recently Gubeladze [Gub09] has shown that any lattice polytope with sufficiently long edges (depending on the dimension) is integrally closed. In view of this result, if there exists a counterexample, it is more likely to be a small polytope. Yet, all polytopes in our classification up to 12 lattice points satisfy the even strongest of these conjectures (see Theorem 23).

Related Results. Let us briefly give an overview over related classification results. Most of them concern toric Fano varieties. In this case, the primitive ray generators of the corresponding fans form the vertices of a convex polytope. In dimension two, $\mathbb{Q}$-Gorenstein toric Fano surfaces are known up to Gorenstein index 17 [KKN10]. In dimension three, the finite list of canonical toric Fano varieties was obtained by A. Kasprzyk [Kas06]. We refer the interested reader to the Graded Ring Database `grdb.lboro.ac.uk` for these and other classification results. Gorenstein toric Fano varieties, corresponding to so-called reflexive polytopes [Bat94], are completely classified up to dimension four [KS98, KS00]. Toric Fano manifolds are classified up to dimension 8 [Bat99, Sat00, KN09, Øbr07]; recently, B. Lorenz computed dimension 9. Smooth reflexive polytopes up to dimension 9 can be found in the data base at `www.polymake.de`. Higher-dimensional classification results of toric varieties are only known in two cases: in the Gorenstein Fano case under strong symmetry assumptions [VK85, Ewa96, Nil06] or if the Picard number of a toric manifold is at most 3, i.e., the d-dimensional fan has at most $d+3$ rays, in which case the variety is automatically projective [KS91, Bat91].

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2. Polarized toric varieties and lattice polytopes.

In this section we introduce notation and recall some basic facts about toric varieties. For more details we refer to [CLS10, §2.3] or [Ful93, Section 3.4].

2.1. **Line bundles and polytopes.** Let k be an arbitrary field. Let $N \cong \mathbb{Z}^d$ be a lattice, and let Σ be a fan of dimension d in $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$. Let $X = X(\Sigma)$ be the associated toric variety, a normal equivariant compactification of the algebraic torus $T \cong (k^*)^d$. The lattice $M = \mathrm{Hom}(N, \mathbb{Z})$ is naturally isomorphic to the character lattice of T. Assume that X is projective, and let $\mathcal{L}$ be an ample line bundle on X. The polarized toric variety $(X, \mathcal{L})$ corresponds to a lattice polytope $P \subseteq M \otimes_{\mathbb{Z}} \mathbb{R}$ of dimension d such that the normal fan to P is equal to Σ. Moreover, we have an isomorphism

$$H^0(X, \mathcal{L}) \cong \bigoplus_{u \in P \cap M} k\chi^u,$$

where $\chi^u \colon T \to k^*, (t_1, \dots, t_d) \mapsto t_1^{u_1} \cdots t_d^{u_d}$ is the character corresponding to $u \in M$. A linear series $W \subseteq H^0(X, \mathcal{L})$ induces a rational map $X \dashrightarrow \mathbb{P}(W)$, which is equivariant if and only if W is torus invariant, that is, $W \cong \oplus_{u \in S} k\chi^u$ for some $S \subseteq P \cap M$. Enumerating the elements in S, the induced map is given by $x \mapsto [\chi^{u_1}(x) \colon \cdots \colon \chi^{u_{\dim W}}(x)]$. Note that the map is induced by a *complete linear series* W if and only if $W = H^0(X, \mathcal{L})$, that is, $S = P \cap M$. See [CLS10, §6]. The degree of this map turns out to be the normalized volume of P – the volume measured in volumes of unimodular simplices.

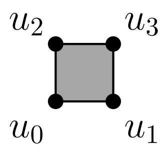

Figure 1: The Segré embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ via $\mathcal{O}(1,1)$

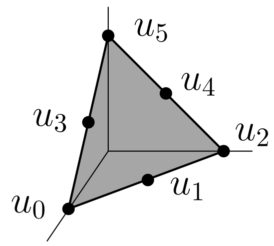

Figure 2: The Veronese embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ via $\mathcal{O}(2)$

For a subset S of $\mathbb{R}^d$, let $\mathrm{aff}(S)$ denote the affine span of S.

Definition 3. We say that two lattice polytopes $P \subset \mathbb{R}^d$ and $P' \subset \mathbb{R}^{d'}$ are *integrally equivalent* if there is a lattice preserving affine map $\mathrm{aff}\, P \to \mathrm{aff}\, P'$ that maps $\mathbb{Z}^d \cap \mathrm{aff}\, P$ bijectively to $\mathbb{Z}^{d'} \cap \mathrm{aff}\, P'$ and P to P'.

Up to this lattice equivalence, we can always assume that our polytope P is full dimensional. Moreover, if P and P' are equivalent, and

if $(X, \mathcal{L})$ and $(X', \mathcal{L}')$ are the corresponding polarized toric varieties, then there exists a torus equivariant isomorphism $\phi\colon X \to X'$ such that $\phi^*\mathcal{L}' \cong \mathcal{L}$.

2.2. **Singularities and cones.** Let $\mathcal{L}$ be an ample line bundle on the toric variety $X(\Sigma)$ with corresponding lattice polytope P. Then $X(\Sigma)$ is covered by torus invariant affine pieces U_u which correspond to the vertices u of P. For a vertex u of P, let $T_uP = \operatorname{cone}(u' - u \mid u' \in P)$ be the *tangent cone* to P at u. It is dual to the *normal cone* $\sigma(P, u)$ of P at u. The semigroup $T_uP \cap M$ is finitely generated. The unique minimal set of generators $\operatorname{Hilb}(T_uP)$ for this semigroup is called the *Hilbert basis* of the cone [CLS10, Proposition 1.2.22]. The coordinate ring of the affine variety U_u is the semigroup ring $k[U_u] = k[T_uP \cap M]$.

The line bundle $\mathcal{L}$ is called *very ample* if its global sections induce an embedding into projective space. The combinatorial condition for $\mathcal{L}$ to be very ample is that for every vertex u of P, the shifted polytope $P - u$ contains the Hilbert basis, i.e., $\operatorname{Hilb}(T_uP) \subseteq P - u$, see [Ful93, Section 3.4].

2.2.1. $\mathbb{Q}$-*Gorenstein cones.* Let $\sigma \subset N_\mathbb{R}$ be a pointed rational d-cone with primitive generators $v_1, \ldots, v_r$. Call σ $\mathbb{Q}$-*Gorenstein* if the v_i lie in a hyperplane inside the linear span $\sigma - \sigma$. In this case, there is a well defined linear functional called *height* $\operatorname{ht}_\sigma \in M_\mathbb{R}$ on σ which takes the value 1 on all v_i. The *index* of σ is the smallest $k \in \mathbb{Z}_{>0}$ so that $k \cdot \operatorname{ht}_\sigma \in M$. We call σ *Gorenstein* if this index is equal to 1.

These notions agree with the notions ($\mathbb{Q}$-)Gorenstein and index for the toric singularity associated with σ. We define the *multiplicity* $\operatorname{mult}(\sigma)$ as the normalized volume of the *nib* of σ

$$\operatorname{nib}(\sigma) := \operatorname{conv}(0, v_1, \ldots, v_r) = \{x \in \sigma \mid \langle \operatorname{ht}_\sigma, x\rangle \leq 1\}$$

which equals the product of the index with the normalized volume of $\operatorname{conv}(v_1, \ldots, v_r)$.

Let P be a lattice polytope with $\mathbb{Q}$-Gorenstein normal fan. We define the *multiplicity* of P to be

$$\operatorname{mult}(P) = \max_u \operatorname{mult}(\sigma(P, u)),$$

the maximal multiplicity of a normal cone to P.

Note that for a toric variety X, the multiplicity does not depend on the polarization, so we can define the multiplicity $\operatorname{mult}(X) = \operatorname{mult}(P)$, where P is a lattice polytope corresponding to an ample line bundle on X.

2.2.2. *Simplicial cones.* The toric singularity U_u is $\mathbb{Q}$-factorial if the tangent cone T_uP of P at u is simplicial, that is, it is generated by a linearly independent set $\{v_1, \ldots, v_d\}$ of primitive vectors. In this case, the singularity U_u is a quotient k^d/G of affine space by a finite abelian group, and the multiplicity is the cardinality of that group.

Such cones are automatically $\mathbb{Q}$-Gorenstein. The box of T_uP is the half open parallelepiped

$$\square(T_uP) := \left\{ \sum_{i=1}^{d} \lambda_i v_i \mid \lambda_i \in [0,1) \text{ for } i = 1, \ldots, d \right\},$$

and a box point is one of the $\operatorname{mult}(T_uP)$ many lattice points in $\square(T_uP)$. Every Hilbert basis element that is not one of the generators of T_uP is a box point, and has smaller height than d. In particular, we have $\operatorname{Hilb}(T_uP) \smallsetminus \{v_1, \ldots, v_d\} \subset \square(T_uP)$.

A cone is called *unimodular* if its primitive minimal generators form a lattice basis. Unimodularity is equivalent to having multiplicity 1.

Definition 4. We call a lattice polytope P *smooth* if every cone in its normal fan is unimodular.

A lattice polytope is smooth if and only if the if the associated projective toric variety X is smooth (see for example [Ful93, Section 2.1]). Moreover, every ample line bundle on a smooth toric variety is very ample.

3. Finiteness Theorems

Before we embark on a proof of our main theorem, let us recall some known finiteness theorems from the literature. The first such theorem goes back to Hensley [Hen83]. The currently best bound is due to Pikhurko [Pik01, (9)].

Theorem 5. *For a positive integer d, there is a bound $V(d)$ so that the volume of every lattice d-polytope with $k \geq 1$ interior lattice points is bounded by $k \cdot V(d)$.*

The second result, due to Lagarias and Ziegler [LZ91, Theorem 2], shows that it is enough to bound the volume if we want to show that only finitely many lattice polytopes satisfy our conditions.

Theorem 6. *A family of lattice d-polytopes with bounded volume contains only a finite number of equivalence classes.*

Corollary 7. *Any family of lattice polytopes with bounded number of lattice points contains only finitely many equivalence classes of polytopes with interior lattice points.*

This explains why the polytopes in examples 15 and 16 do not have interior lattice points. Here is now our first main result.

Theorem 8. *For a nonnegative integer d and a finite family $\mathcal{F}$ of rational d-cones, there are only finitely many lattice d-polytopes with $N+1$ lattice points all whose normal cones are equivalent to cones belonging to $\mathcal{F}$.*

We first prove that this is true in dimension two.

Lemma 9. *For a finite family $\mathcal{F}$ of rational 2-dimensional cones, there are only finitely many lattice polygons with $N+1$ lattice points all whose normal cones are equivalent to cones belonging to $\mathcal{F}$.*

Proof. Let P be a lattice polygon, and let u be a vertex of P with normal cone σ. Up to equivalence, we can assume $u = 0$ and $\sigma = \mathrm{cone}[(0,1),(p,q)]$ with $0 \leq q < p$. Because σ is equivalent to a cone in the finite family $\mathcal{F}$, the multiplicity p is bounded, and there are only finitely many choices for p and q.

Then P has a vertex $u' = (x', 0)$ with normal cone σ' such that $\sigma' = \mathrm{cone}[(0,1),(-p',q')]$ where $p' > 0$. Again, σ' is equivalent to a cone in $\mathcal{F}$, and there are only finitely many choices for p'.

With this notation, P contains $x'+1$ lattice points on the edge uu', and the segment in P parallel to uu' at distance one has length $\ell := x' + q'/p' + q/p$, and thus contains more than $\ell - 1$ lattice points. Now the inequalities $x' \geq 1$, $\ell \geq 0$ and $x' + \ell \leq N+1$ together leave only finitely many choices for x' and q'.

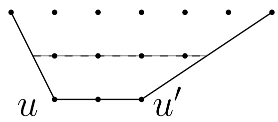

Figure 3: Lots of lattice points at distance one.

The normal fan together with all edge lengths determines P up to equivalence. ☐

Proof of Theorem 8. The number of combinatorial types of fans with $\leq N+1$ maximal cones is finite. Here, the combinatorial type is given by the set of faces, partially ordered by inclusion. For each combinatorial type there are only finitely many choices of cones from $\mathcal{F}$ for its maximal faces. So we can restrict to a class of normal fans with a fixed combinatorial type and with given equivalence class for each maximal face.

For instance, in dimension two, the fans of Hirzebruch surfaces (compare Example 17) all have the same combinatorial type and all maximal faces of the fan are unimodular.

By Lemma 9, we can also fix the integral type for every 2-face of P. We claim that these choices determine P up to equivalence.

To this end, fix a vertex u of P with normal cone $\sigma \in \mathcal{F}$. This determines all 2-dimensional faces of P incident to u. In particular, if u' is another vertex of P adjacent to u, then u' together with all edges which are incident to u' and contained in a common 2-face with u are

determined. The directions of these edges together with the edge uu' span $\mathbb{R}^d$ as a vector space. They thus pin down the normal cone at u'.

In summary, fixing a vertex and its tangent cone also fixes all adjacent vertices and their tangent cones. As the vertex-edge graph of P is connected, this determines P. □

Corollary 10. *For nonnegative integers d and $N+1$, there are only finitely many equivalence classes of smooth d-polytopes with $N+1$ lattice points.*

Proof. In the case of smooth polytopes, $\mathcal{F}$ contains only one cone: the standard orthant. □

Corollary 11. *For nonnegative integers d, m and N, there are only finitely many d-polytopes with $\mathbb{Q}$-Gorenstein normal cones of multiplicity bounded by m and with $N+1$ lattice points.*

Proof. Applying Theorem 6 to the convex hull of 0 and the primitive generators of a $\mathbb{Q}$-Gorenstein cone, we see that the family of $\mathbb{Q}$-Gorenstein cones with multiplicity $\leq m$ contains only finitely many equivalence classes. □

Using the dictionary between toric morphisms and lattice polytopes, Corollary 11 implies the following corollary. We consider two morphisms to $\mathbb{P}^N$ the same if they differ by an automorphism of $\mathbb{P}^N$.

Corollary 12. *Let N and m be nonnegative integers. There are finitely many morphisms from some $\mathbb{Q}$-Gorenstein toric variety X of dimension less than N with $\operatorname{mult}(X) \leq m$ to $\mathbb{P}^N$ that are induced by a complete linear series.*

In order to deduce Theorem 1 from Corollary 12, we need another lemma.

Lemma 13. *Let N and d be nonnegative integers. Then there are up to equivalence only finitely many $\mathbb{Q}$-Gorenstein cones $\sigma \subset \mathbb{R}^d$ so that*

$$(*) \qquad \#\operatorname{Hilb}(\sigma) \ + \ \#(\operatorname{nib}(\sigma) \cap \mathbb{Z}^d) \ \leq \ N+1\,.$$

Proof. We will show by induction on d that $(*)$ implies that $\operatorname{mult}(\sigma)$ is bounded. Then Theorem 6 implies that there are only finitely many choices for σ.

For $d=1$ there is only one cone. For $d=2$, Pick's formula [Pic99] tells us that $\operatorname{mult}(\sigma) \leq 2\#(\operatorname{nib}(\sigma) \cap \mathbb{Z}^2) - 5$. So let us assume that the lemma is true for $d-1$. Because of Corollary 7, we can assume that $\operatorname{nib}(\sigma)$ has no interior lattice points. This implies that all interior Hilbert basis elements of σ have height ≥ 1. By induction, there is a minimal height $\epsilon(d-1,N) > 0$, depending only on $d-1$ and N, of a Hilbert basis element in the boundary of σ. Let $\epsilon = \min\{\epsilon(d-1,N), 1\}$.

Triangulate $\sigma = \cup_{i=1}^r \sigma_i$ into simplicial cones using only rays of σ. Every Hilbert basis element of σ is a box point of one of the σ_i. As σ has at most N rays, every box point belongs to less than $\binom{N}{d}$ of the σ_i.

Now, every box point of every σ_i has a representation $\sum_{v\in\mathrm{Hilb}(\sigma)} a_v v$ with $a_v \in \mathbb{Z}_{\geq 0}$. On the other hand, any box point has height $< d$, so that in the above representation we must have $\epsilon \cdot \sum_{v\in\mathrm{Hilb}(\sigma)} a_v < d$ which leaves at most $\binom{N+\lfloor d/\epsilon\rfloor}{N}$ possibilities. In other words,

$$\mathrm{mult}(\sigma) = \sum_{i=1}^r \#\Box(\sigma_i) < \binom{N}{d} \cdot \# \left(\bigcup_{i=1}^r \Box(\sigma_i)\right) < \binom{N}{d}\binom{N+\lfloor d/\epsilon\rfloor}{N}.$$

□

Proof of Theorem 1. Since X is $\mathbb{Q}$-factorial, P is simple. This means that every tangent cone to P is simplicial, in particular, $\mathbb{Q}$-Gorenstein. Moreover, since P corresponds to a very ample line bundle, a translate of the Hilbert basis for each tangent cone is a subset of the lattice points of P. Since P has $N+1$ lattice points, it follows from Lemma 13 that there are only finitely many equivalence classes of tangent cones. So there are only finitely many equivalence classes of normal cones. Now the claim follows from Theorem 8. □

Remark 14. If we require the embedding of X given by P to be projectively normal, then the homogeneous coordinate ring is a Cohen-Macaulay standard graded algebra [Hoc72] with $\leq N+1$ generators. Thus, its Hilbert function (the Ehrhart series of P) is bounded (compare, e.g. [Hib92, Lemma 18.1]). This bounds the degree (the normalized volume of P). By Theorem 6, there are only finitely many such P.

The following examples show that we need to assume that X is $\mathbb{Q}$-factorial in Theorem 1 (Example 15) and that the multiplicities are bounded in Corollary 11/12 (Example 16).

Example 15. In [MFO07, p.2290] Winfried Bruns gives an example of a very ample divisor on a toric 3-fold whose complete linear series does not yield a projectively normal embedding. This example generalizes to a family of very ample polytopes

$$Q_k := \mathrm{conv}\begin{pmatrix} 0&1&0&0&1&0&1&1\\ 0&0&1&0&0&1&1&1\\ 0&0&0&1&1&1&k&k+1\end{pmatrix}$$

with 8 lattice points but unbounded volume. Observe that these polytopes have a Gorenstein normal fan with $\mathrm{mult}(Q_k) = k+1$. However, $T_{(0,1,0)}(Q_k) = \mathrm{cone}((0,-1,0),(0,0,1),(1,-1,0),(1,0,k))$ is not Gorenstein for $k \geq 2$.

Example 16. In [Ree57] John Reeve constructs the simplices

$$P_k = \mathrm{conv}\begin{pmatrix} 0&1&0&1\\ 0&0&1&1\\ 0&0&0&k\end{pmatrix}$$

with 4 lattice points but unbounded volume. In particular, this family of polytopes shows that the number of lattice points of a lattice polytope does not give a bound on its volume. On the other hand, note that the line bundle corresponding to P_k is not very ample. The line bundle corresponding to $2P_k$ is normally generated, so in particular it is very ample (see [BGT97, Theorem 1.3.3] or [ON02]). It induces an embedding into $\mathbb{P}^{k+8}$.

The finiteness theorem does not mean that there are only finitely many projective torus equivariant embeddings into a fixed projective space.

Example 17. Figure 4 shows how to embed an arbitrary Hirzebruch surface torically into $\mathbb{P}^5$.

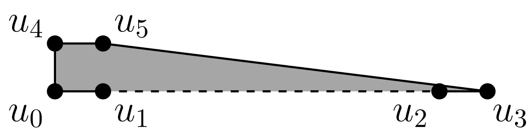

Figure 4: Hirzebruch surface $\hookrightarrow \mathbb{P}^5$

Corollary 10 can also be strengthened.

Theorem 18. *For nonnegative integers d and N, there are only finitely many equivalence classes of smooth d-polytopes with $N+1$ lattice points on edges.*

Again, the proof relies on the study of the two-dimensional situation.

Lemma 19. *There are only finitely many smooth polygons with $N+1$ lattice points on the boundary. Moreover,*

$$\mathrm{vol}(P) \leq \begin{cases} \frac{1}{2}b^2 & , \quad a=3 \\ b^2 & , \quad a=4 \\ 4^{a-4}b^2 - a + 4 & , \quad a \geq 5 \end{cases},$$

where a denotes the number of vertices and b the length of a maximal edge.

Example 20. Here is an example, due to Paco Santos, with all edges of length one whose volume grows exponentially with the number of vertices. This shows that the upper bound $4^a b^2$ above is actually not that bad. The construction gives a lower bound of $1.272^a b^2$.

Let F_n denote the n-th Fibonacci number. That is, $F_0 = 0$, $F_1 = F_2 = 1$, $F_3 = 2$, etc.

We will use the following well known expressions, which say that certain pairs of vectors form unimodular cones. The sign of the right

hand sides depends on the parity of indices, and will be important later:

$$F_{2k+3}F_{2k} - F_{2k+2}F_{2k+1} = -1$$

$$F_{2k+2}F_{2k-1} - F_{2k+1}F_{2k} = +1$$

$$F_{2k+4}F_{2k} - F_{2k+2}^2 = -1$$

$$F_{2k+3}F_{2k-1} - F_{2k+1}^2 = +1$$

Then, start constructing the boundary of the polygon by concatenating the $2k+1$ edges with vector $(F_2, F_0) = (1, 0)$, (F_4, F_2), (F_6, F_4),. . . , (F_{2k}, F_{2k-2}), (F_{2k-1}, F_{2k-3}), (F_{2k-3}, F_{2k-5}), . . . , $(F_3, F_1) = (2, 1)$, $(1, 1)$, $(0, 1)$.

Every consecutive pair of segments is unimodular, because of the above relations. The signs in the right-hand sides say that the chain is convex. Since the chain starts with $(1, 0)$ and ends with $(0, 1)$, one can build a convex smooth polygon by reflecting it horizontally and vertically. This polygon has $a = 8k + 4$ vertices and volume $\Theta(F_{2k+2}F_{2k}) = \Theta(\tau^{4k})$ where $\tau = (1 + \sqrt{5})/2$ is the golden ratio. The latter comes from the fact that the sum of the first $k-2$ Fibonacci numbers equals (almost) the k-th Fibonacci number. Hence, the polygon is inscribed in a rectangle of sides about $2F_{2k+2}$ and $2F_{2k}$, and it contains the mid-points of the edges of this rectangle, so it contains at least half of the rectangle. Since the Fibonacci sequence grows exponentially, we are done.

For example, for $k = 5$ the basic chain is: $(1, 0)$, $(3, 1)$, $(8, 3)$, $(21, 8)$, $(55, 21)$, $(34, 13)$, $(13, 5)$, $(5, 2)$, $(2, 1)$, $(1, 1)$, $(0, 1)$.

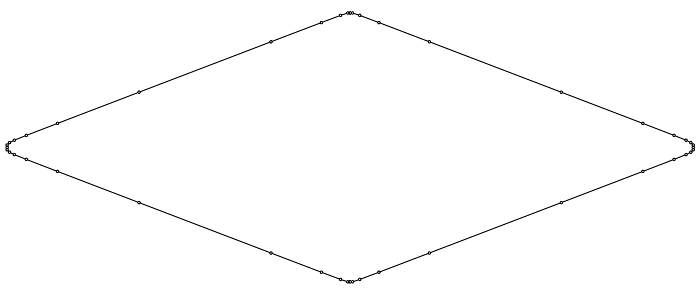

Figure 5: The 5^{th} Fibonacci polygon

This example demonstrates drastically that the sequence of edge lengths does not determine the area. On the other end of the range, there are the polygons of Imre Bárány and Norihide Tokushige [BT04,

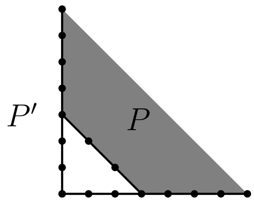

Figure 6: P' is given by "blowing down" P.

Rk.2] which are also smooth, also have edge length sequence $(1, \dots, 1)$ but have area less than $a^3/54$.

Proof of Lemma 19. By Theorem 6 we only have to show the inequalities. For this, we induct on the number of vertices a.

From the classification of two-dimensional complete unimodular fans (compare [Ewa96, Theorem 6.6]) we can easily check the statement for $a \leq 4$ (here the fan corresponds either to the projective plane or a Hirzebruch surface).

For $a \geq 5$, there exist three consecutive rays in the fan such that the corresponding primitive lattice points v_{i-1}, v_i, v_{i+1} satisfy $v_{i-1} + v_{i+1} = v_i$ (see again [Ewa96, Theorem 6.6]). As Figure 6 shows this implies that there is a smooth polygon P' containing P and having $a - 1$ vertices whose maximal edge length equals at most $2b$. This yields the statement by induction. More precisely, the length of the edge blown down equals the difference in boundary lattice points. □

The upper bound can be improved to $2^a b^2$. The idea is that there has to be more than one edge that can be "blown down". In fact, there must always be at least three and (as soon as $a > 6$) one can always find two of them "at distance at least three", meaning that the two are incident to four other edges in total. Those two edges can be blown down simultaneously to get a polygon with $a-2$ vertices and maximum length at most $2b$.

Proof of Theorem 18. Since P contains only a bounded number of vertices, there are only finitely many possible combinatorial types for P. Let us fix a vertex u with unimodular normal cone $\sigma = \operatorname{cone}(v_1, \dots, v_d)$. By Lemma 19, there are only finitely many choices for the $\binom{d}{2}$ many 2-faces incident to u. Any such choice determines the positions of all vertices of P adjacent to u as well as their normal cones. Since the vertex-edge-graph of P is connected, this finishes the proof. □

4. Classification in Dimension 3

This section summarizes the strategy to classify smooth 3-polytopes with at most 12 lattice points, implementing the proof of Corollary 10. For full details, including source code, see [Lor09, HLP10].

4.1. **Generating Normal Fans.** Katsuya Miyake and Tadao Oda have classified smooth 3-dimensional fans which are minimal with respect to equivariant blow-ups [Oda88, Theorem 1.34]. This classification goes up to at most eight rays or equivalently, 12 full-dimensional cones. Starting from this list, all possible sequences of blow-ups had to be enumerated until no fan of a polytope with ≤ 12 lattice points could occur further down the search tree. In order to prune the search tree, we used bounds based on the two-dimensional classification.

4.2. **Generating Polytopes.** The next step is to find the polytopes, given the normal fan Σ. We write the primitive generators of the rays of Σ as rows of the matrix A, and describe the polytopes with inequalities

$$P(A, b) = \{x \mid Ax \leq b\} .$$

The edge lengths of $P(A, b)$ depend linearly on the right-hand side vector b – provided we stay inside the chamber of all b yielding a polytope with normal fan Σ (the nef-cone of the toric variety $X(\Sigma)$). If we require these edge length functionals to be positive, we obtain an inequality description of the chamber. Bounding the sum of the edge lengths, the search space of possible b-vectors which yield at most $N+1$ lattice points becomes itself the set of lattice points in a polytope.

The last step is to remove all polytopes that are lattice-isomorphic to another one in the list.

All these computations can be done with the `polymake` lattice polytope package by Benjamin Lorenz, Andreas Paffenholz and Michael Joswig [GJ, JMP09] using interfaces to `4ti2` by the 4ti2 team [HH03], `Latte` by Jesús De Loera et.al. [LHTY04, LHTY] and `normaliz2` by Winfried Bruns et al. [BK01, BIS].

4.3. **Classification Results.**

Theorem 21. *There are 41 equivalence classes of smooth lattice polygons with at most 12 lattice points.*

Vertices	*3*	*4*	*5*	*6*	*7*	*8*	*≥9*
Polygons	*3*	*30*	*3*	*4*	*0*	*1*	*0*

Theorem 22. *There are 33 equivalence classes of smooth three-dimensional lattice polytopes with at most 12 lattice points.*

Vertices	*4*	*6*	*8*	*≥10*
Polytopes	*2*	*25*	*6*	*0*

Lists of all smooth polygons and smooth 3-polytopes with at most 12 lattice points can be found in the appendix.

4.4. **Comments.** We now have a list of smooth lattice polytopes in dimensions two and three with at most 12 lattice points. The bound 12 may seem rather low – the smallest 3-polytope with one interior lattice point has 21 lattice points total [Kas08]. The classification carried out

here serves as a proof of concept – it can be done. There are several points in the algorithm where it could be improved (compare [Lun10]). One easily proves by inspection of the list in the appendix that all these polytopes satisfy all the ideal theoretic conjectures mentioned in the introduction. In particular, the homogeneous coordinate ring is a Koszul algebra.

Theorem 23. *If P is a 3-dimensional smooth polytope with at most* 12 *lattice points, then P has a regular unimodular triangulation with minimal non-faces of size two.*

Corollary 24. *Let X be a toric threefold embedded in $\mathbb{P}^{\leq 11}$ using a complete linear series. Then the defining ideal of X has an initial ideal generated by square-free quadratic monomials.*

In the current implementation, the generation of the normal fans is the bottleneck. By implementing a different way to directly generate all smooth normal fans one could skip the big recursion calculating all blowing-ups, as well as overcome the limits of at most 12 vertices imposed by the Miyake/Oda classification. The second point to work on is the calculation of lattice points of the polytope containing all right-hand sides b. The dimension of this polytope is equal to the Picard number of the toric variety: the number of rays of the fan minus the ambient dimension. Of course, better theoretical bounds for all steps of the algorithm will directly impact the performance.

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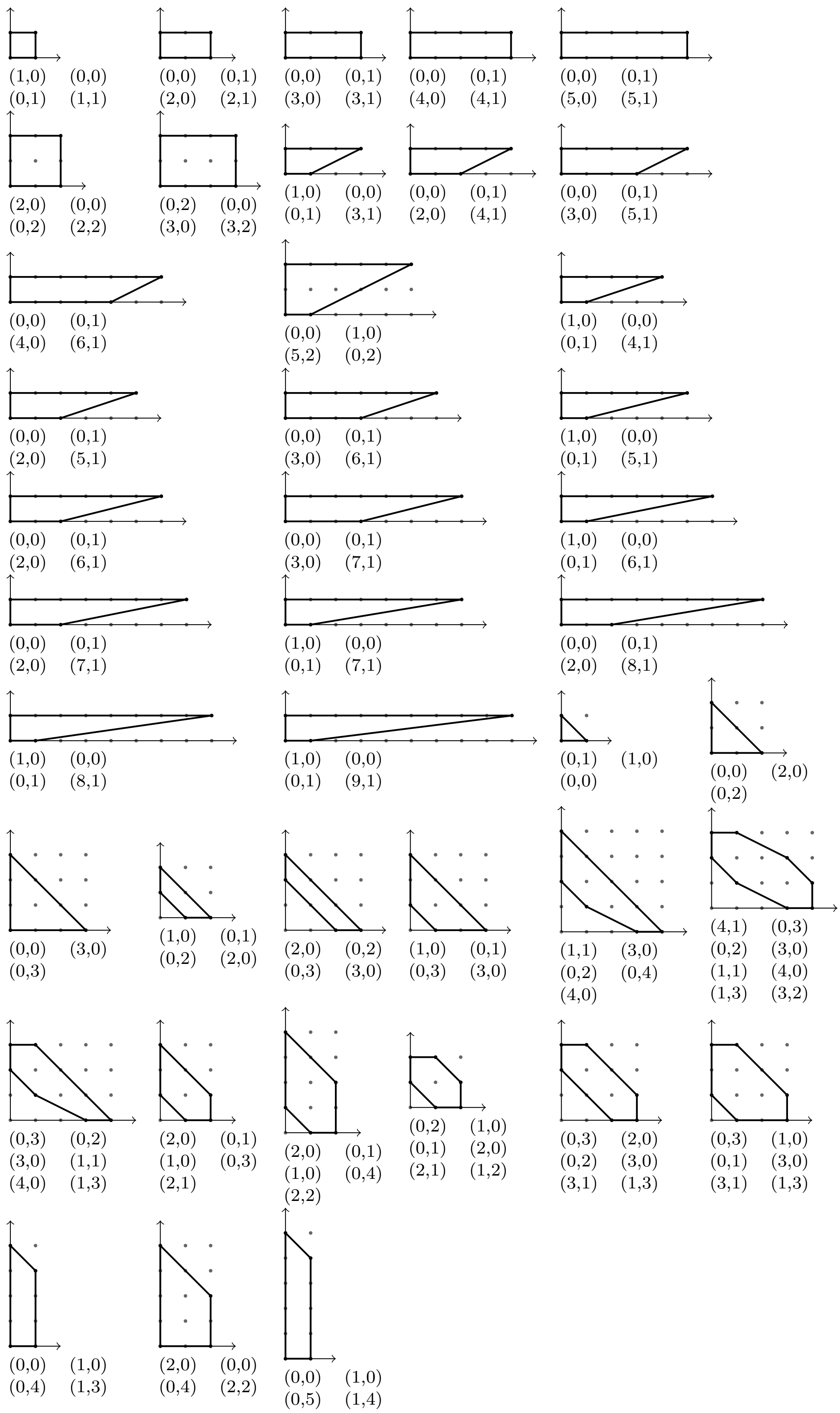
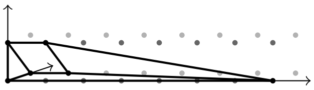
List of Smooth Polygons with ≤ 12 Lattice Points



List of Smooth 3-Polytopes with ≤ 12 Lattice Points



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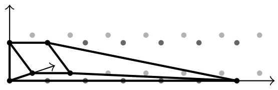

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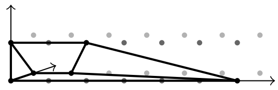

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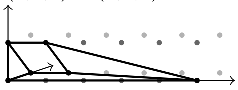

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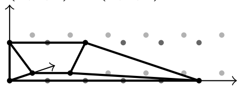

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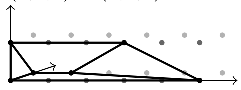

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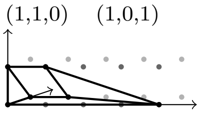

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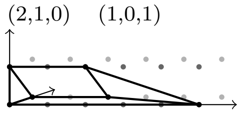

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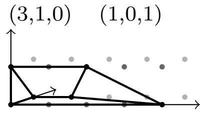

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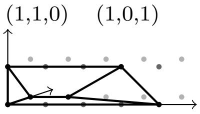

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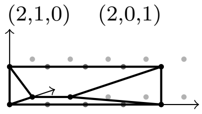

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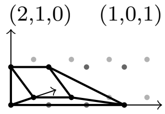

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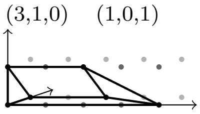

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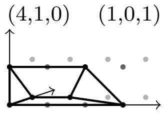

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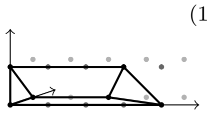

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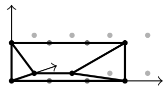

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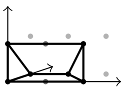

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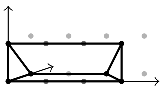

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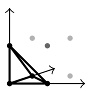

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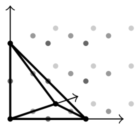

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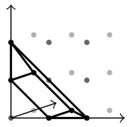

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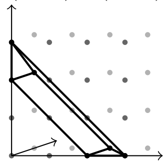

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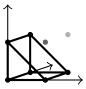

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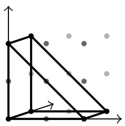

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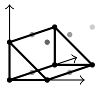

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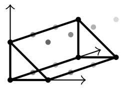

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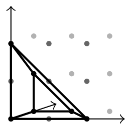

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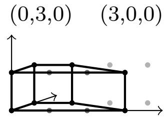

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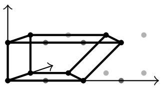

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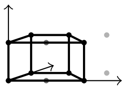
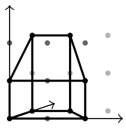
(1,0,1) (2,1,1)



(0,1,0) (0,0,0)
(0,0,1) (0,1,1)
(2,0,0) (2,1,0)
(1,0,1) (1,1,1)



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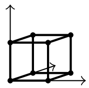

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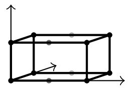
(0,1,1) (1,1,1)



(0,1,0) (0,0,0)
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